

# Toroidal $p$ -branes, anharmonic oscillators and (hyper)elliptic solutions

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## Abstract

Exact solvability of brane equations is studied, and a new  $U(1) \times U(1) \times \dots \times U(1)$  invariant ansatz for the solution of  $p$ -brane equations in  $D = (2p+1)$ -dimensional Minkowski space is proposed. The reduction of the  $p$ -brane Hamiltonian to the Hamiltonian of  $p$ -dimensional relativistic anharmonic oscillator with the monomial potential of the degree equal to  $2p$  is revealed. For the case of degenerate  $p$ -torus with equal radii it is shown that the  $p$ -brane equations are integrable and their solutions are expressed in terms of elliptic ( $p = 2$ ) or hyperelliptic ( $p > 2$ ) functions. The solution describes contracting  $p$ -brane with the contraction time depending on  $p$  and the brane energy density. The toroidal brane elasticity is found to break down linear Hooke law as it takes place for the anharmonic elasticity of smectic liquid crystals.

## 1 Introduction

Membranes and  $p$ -branes play an important role in M/string theory [1], and their quantization is one of current hot problems. Their solution is compli-

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cated by nonlinearity of the classical brane equations in contrast to the string ( $p=1$ ) case [2]. The question of the membrane ( $p=2$ ) and  $p$ -brane dynamics and its integrability attracts much attention (see e.g. [3-15]). P-branes belong to Hamiltonian dynamical systems for which there is known the notion of complete integrability in the Liouville sense. The notion implies the existence of a maximal set of functions on the phase space which have zero Poisson brackets between themselves and with the system Hamiltonian. These functions are time-independent and split the whole phase space into hypersurfaces which are closed in the process of the system evolution. A finite dimensional phase space has the dimension equal to  $2n$  and the maximal number of such independent invariants is  $n$ , respectively. If a Hamiltonian  $H$  and associated PB's do not explicitly depend on time then  $H$  belongs to one of the mentioned invariants, and the system preserves its energy coinciding with  $H$ . For the case of compact energy levels the invariant hypersurfaces of completely integrable systems may be transformed into two-dimensional tori associated with special sets of canonical pairs called the action and angle variables [16]. The action variables are preserved integrals constructed by integration of the one-forms  $pdq$  along the cycles of invariant tori on which the angle variables have the sense of angle parameters. The Hamiltonian equations for these angle variables turn out to be linear. Thus, the question of complete integrability of general Hamiltonian equations may be reformulated into the equivalent problem of an explicit construction of the angle action variables. The Hamilton-Jacobi method helps to construct such variables using special canonical transformations which result in a new Hamiltonian independent of all generalized coordinates. Then the integrability problem is reduced to the construction of complete solution of the corresponding Hamilton-Jacobi equation. P-branes are field systems with an infinite number of the degrees of freedom, because their world vectors  $x_m$  depend on continuous world-volume parameters. Generalization of the notion of complete Liouville integrability for Hamiltonian nonlinear PDE's implies the existence of an infinite number of Poisson commuting invariants. The sine-Gordon, KdV and KP equations give well known examples of completely integrable equations in two and three dimensions, respectively. In addition to the existence of an infinite number of conservation laws these equations are characterized by the presence of Backlund transformations and solitonic solutions which describe non-perturbative sectors in the space of solutions. The Backlund transformations permit to generate one-parametric family of new solutions starting from one known partial soliton. As a result, the given integrals of motion become dependent

on the Backlund parameter, and their expansion in this parameter yields an infinite number of new conservation laws. The Backlund parameter is connected with the time-independent eigenvalue of isospectral linear eigenvalue problem which plays a fundamental role in the inverse scattering method. A key role in the proof of integrability belongs to so called Lax pair of some linear operators such that their Poisson bracket reproduces the nonlinear equation under investigation (see e.g. [17]). This gives a general background for discussing the  $p$ -brane integrability problem.

A direct application of the inverse spectral method to  $p$ -brane equations meets essential problems, but there is some progress in study of various particular solutions of the equations. However, not so much is known about such solutions. While studying this problem Hoppe proposed the  $U(1)$  invariant ansatz for closed membranes and reformulated their equations in  $D=5$  into the system of 2-dim nonlinear equations [18]. The elliptic solution of these equations, describing a family of closed contracting 2d tori, together with the solution corresponding to a spinning 2d torus, found in [19] give an example of time-dependent particular solutions. The static equations of  $U(1)$  invariant membranes in  $D = (2N + 1)$ -dimensional Minkowski space were integrated and their general solution for any  $N > 1$  was presented in [19]. The geometric approach to the description of the invariant membranes in  $D=5$  developed in [20] revealed their connection with the nonlinear pendulum equation and a two-dimensional generalization of the nonlinear Abel equation.

Here we extend this approach to branes and propose an ansatz for the solution of  $p$ -brane equations in  $D = (2p + 1)$ -dimensional Minkowski space with any integer  $p > 1$ . The strong  $(D, p)$ -correlation between the space-time and brane dimensions covers, in particular, interesting cases of globally invariant 5-branes of M/string theory in  $D=11$  space-time, membranes in  $D=5$  and 3-branes in  $D=7$ . The proposed ansatz corresponds to closed compact  $p$ -branes with the global rotational symmetry  $U(1) \times U(1) \times \dots \times U(1)$  (with  $p$  multipliers) of their  $p$ -dimensional hypersurfaces  $\Sigma_p$ . These hypersurfaces turn out to be isometric to flat  $p$ -tori with zero curvatures. The Hamiltonians and equations of invariant  $p$ -branes are constructed, and it is shown that they describe  $p$ -dimensional anharmonic oscillators with the quartic potential for membranes in  $D=5$ , and the monomial potential of the degree  $2p$  for the  $p$ -brane in  $D=2p+1$ . A characteristic feature of these Hamiltonians is the absence of the quadratic terms which are contained in the Hamiltonian

of the harmonic oscillator<sup>1</sup>. The p-brane equations are reformulated into the equations for an elastic media with the symmetric stress tensor corresponding to isotropic pressure. For the case of degenerate p-torus with all equal radii these nonlinear equations are integrable and their solutions are expressed in terms of the elliptic cosine for  $p = 2$  or hyperelliptic functions for higher  $p > 2$ . The solutions describe contracting  $p$ -branes with the contraction time depending on their dimension  $p$  and energy density. The dependence of  $p$  is partially localized in the Euler beta function. We revealed that an elastic medium, associated with the  $p$ -toroidal branes, is characterized by a nonlinear elasticity law which generalizes the linear Hooke law. The brane elasticity is found to be similar to the anharmonic elasticity earlier discovered in the statistical physics of 2D and 3D smectic liquid crystals [21], [22], [23].

## 2 P-brane dynamics for any $(D, p)$

The Dirac action for a p-brane without boundaries is defined by the integral in the dimensionless worldvolume parameters  $\xi^\alpha$  ( $\alpha = 0, \dots, p$ )<sup>2</sup>

$$S = T \int \sqrt{|G|} d^{p+1} \xi,$$

where  $G$  is the determinant of the induced metric  $G_{\alpha\beta} := \partial_\alpha x_m \partial_\beta x^m$  and  $T$  is the p-brane tension with the dimension  $L^{-(p+1)}$ , because  $x^m$  has the dimension of length. After splitting of the  $p$ -brane world vector  $x^m = (x^0, x^i) = (t, \vec{x})$  and internal coordinates  $\xi^\alpha = (\tau, \sigma^r)$ , the Euler-Lagrange equations and  $(p+1)$  primary constraints generated by  $S$  take the form

$$\partial_\tau \mathcal{P}^m = -T \partial_r (\sqrt{|G|} G^{r\alpha} \partial_\alpha x^m), \quad \mathcal{P}^m = T \sqrt{|G|} G^{\tau\beta} \partial_\beta x^m, \quad (1)$$

$$\tilde{T}_r := \mathcal{P}^m \partial_r x_m \approx 0, \quad \tilde{U} := \mathcal{P}^m \mathcal{P}_m - T^2 |\det G_{rs}| \approx 0, \quad (2)$$

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<sup>1</sup>Usually the notion of anharmonic oscillator is used in the case when the quartic and higher terms in the potential energy are small in comparison with its quadratic terms. Here we use this term despite the absence of the quadratic term in the p-brane Hamiltonians and do not assume the smallness of the higher monomials.

<sup>2</sup>Here the D-dimensional Minkowski space has the signature  $\eta_{mn} = (+, -, \dots, -)$ .

where  $\mathcal{P}^m$  is the energy-momentum density. Then we use the orthogonal gauge simplifying the metric  $G_{\alpha\beta}$

$$\begin{aligned} L\tau = x^0 \equiv t, \quad G_{\tau r} = -L(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0, \\ g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad G_{\alpha\beta} = \begin{pmatrix} L^2(1 - \dot{\vec{x}}^2) & 0 \\ 0 & -g_{rs} \end{pmatrix} \end{aligned} \quad (3)$$

with  $\dot{\vec{x}} := \partial_t \vec{x} = L^{-1} \partial_\tau \vec{x}$ . The solution of the constraint  $\tilde{U}$  (2) has the form

$$\mathcal{P}_0 = \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}, \quad g = \det(g_{rs}) \quad (4)$$

and becomes the Hamiltonian density  $\mathcal{H}_0$  for the p-brane since  $\dot{\mathcal{P}}_0 = 0$  in view of Eq. (1). Using the definition of  $\mathcal{P}_0$  (1) and  $G^{\tau\tau} = 1/L^2(1 - \dot{\vec{x}}^2)$  we find the expression of  $\mathcal{P}_0$  as a function of the p-brane velocity  $\dot{\vec{x}}$

$$\mathcal{P}_0 := TL\sqrt{|G|G^{\tau\tau}} = T\sqrt{\frac{|g|}{1 - \dot{\vec{x}}^2}}. \quad (5)$$

Taking into account this expression and the definition (1) one can present  $\vec{\mathcal{P}}$  and its evolution equation (1) as

$$\vec{\mathcal{P}} = \mathcal{P}_0 \dot{\vec{x}}, \quad \dot{\vec{\mathcal{P}}} = T^2 \partial_r \left( \frac{|g|}{\mathcal{P}_0} g^{rs} \partial_s \vec{x} \right). \quad (6)$$

Then Eqs. (6) yield the second-order PDE for  $\vec{x}$

$$\ddot{\vec{x}} = \frac{T}{\mathcal{P}_0} \partial_r \left( \frac{T}{\mathcal{P}_0} |g| g^{rs} \partial_s \vec{x} \right). \quad (7)$$

These equations may be presented in the canonical Hamiltonian form

$$\dot{\vec{x}} = \{H_0, \vec{x}\}, \quad \dot{\vec{\mathcal{P}}} = \{H_0, \vec{\mathcal{P}}\}, \quad \{\mathcal{P}_i(\sigma), x_j(\tilde{\sigma})\} = \delta_{ij} \delta^{(p)}(\sigma^r - \tilde{\sigma}^r)$$

using the integrated Hamiltonian density (4)  $\mathcal{H}_0 (= \mathcal{P}_0)$

$$H_0 = \int d^p \sigma \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}. \quad (8)$$

The presence of square root in (8) points out on the presence of the known residual symmetry preserving the orthogonal gauge (3)

$$\tilde{t} = t, \quad \tilde{\sigma}^r = f^r(\sigma^s) \quad (9)$$

and generated by the constraints  $\tilde{T}_r$  (2) reduced to the form

$$T_r := \vec{\mathcal{P}} \partial_r \vec{x} = 0 \quad \Leftrightarrow \quad \dot{\vec{x}} \partial_r \vec{x} = 0, \quad (r = 1, 2, \dots, p). \quad (10)$$

The gauge freedom associated with the symmetry (9) allows to put  $p$  additional time-independent conditions on  $\vec{x}$  and its space-like derivatives.

The above description of brane dynamics is valid for any space-time and brane world-volume dimensions  $(D, p)$  (with  $D > p$ ).

### 3 $U(1) \times U(1) \times \dots \times U(1)$ invariant p-branes

Here we consider p-branes evolving in  $D = (2p + 1)$ -dimensional Minkowski space-time, and assume that their shape is invariant under the direct product  $\mathcal{U} := \prod_{a=1}^p U_a(1)$ . Each of these  $U(1)$  symmetries is locally isomorphic to one of the  $O(2)$  subgroups of the  $SO(2p)$  subgroup of the Euclidean rotations. Thus, the p-dimensional hypersurface  $\Sigma_p$  of  $\mathcal{U}$  invariant p-brane has the group  $\mathcal{U}$  as its isometry with  $p$  Killing vectors. This points to the existence of a parametrization of the p-brane hypersurface  $\Sigma_p$  with the metric tensor  $g_{rs}$  independent of  $\sigma^r$ . Our analysis will be restricted by the case of  $\mathcal{U}$  invariant p-branes without boundaries. The invariant p-branes with boundaries are treated similarly taking into account additional boundary terms.

To construct  $\mathcal{U}$  invariant p-brane hypersurface, we propose the following anzats for its space world vector  $\vec{x}$

$$\begin{aligned} \vec{x}^T &= (q_1 \cos \theta_1, q_1 \sin \theta_1, q_2 \cos \theta_2, q_2 \sin \theta_2, \dots, q_p \cos \theta_p, q_p \sin \theta_p), \\ q_a &= q_a(t), \quad \theta_a = \theta_a(\sigma^r), \quad (a = 1, \dots, p), \end{aligned} \quad (11)$$

where  $T$  is the transposition of the column usually used for a vector components. This space vector lies in the  $2p$ -dimensional Euclidean subspace of  $(2p + 1)$ -dimensional Minkowski space and automatically satisfies the orthogonality constraints (10):  $\dot{\vec{x}} \partial_r \vec{x} = 0$ . The anzats (11) originates from the realization of  $2p$ -dimensional vector  $\vec{x}$  describing any p-brane, and is defined by  $p$  pairs of its "polar" coordinates

$$\vec{x}^T(t, \sigma^r) = (q_1 \cos \theta_1, q_1 \sin \theta_1, \dots, q_p \cos \theta_p, q_p \sin \theta_p) \quad (12)$$

with the coordinates  $q_a, \theta_a$  depending on all the parameters  $(t, \sigma^1, \dots, \sigma^p)$  of the p-brane world volume:  $q_a = q_a(t, \sigma^r)$ ,  $\theta_a = \theta_a(t, \sigma^r)$ .

Thus, the proposed anzats (11) is obtained from the general representation (12) by excluding the time dependence for all "polar" angles  $\theta_a = \theta_a(\sigma^r)$  as well as  $\sigma^r$  dependence for all "radial" coordinates  $q_a = q_a(t)$ . As a result, at any fixed moment  $t$  the vector  $\vec{x}^T$  (11) is produced from the vector  $\vec{x}_0^T = (q_1, 0, q_2, 0, \dots, q_p, 0)$  by the transformations of the diagonal subgroup  $\mathcal{U} \in SO(2p)$ , parametrized by the angles  $\theta_a$  which rotate the planes  $x_1x_2, x_3x_4, \dots, x_{2p-1}x_{2p}$ .

So, the anzats (11) describes one of the representatives of the family of  $\mathcal{U}$  invariant (hyper)surfaces embedded in the  $2p$ -dimensional Euclidean space. Each of the members of the family has the symmetry  $\mathcal{U}$  as its inherent symmetry. The  $p$ -brane worldvolume metric  $G_{\alpha\beta}$  corresponding to (11) has the form similar to (3) with the non-zero components

$$G_{tt} = 1 - \dot{\mathbf{q}}^2, \quad \mathbf{q} := (q_1, \dots, q_p), \quad g_{rs} = \sum_{a=1}^p q_a^2 \theta_{a,r} \theta_{a,s}, \quad (13)$$

where  $\theta_{a,r} \equiv \partial_r \theta_a$ ,  $\dot{\mathbf{q}} \equiv \partial_t \mathbf{q}$ , and yields the following interval  $ds^2$  on  $\Sigma_{p+1}$

$$ds_{p+1}^2 = (1 - \dot{\mathbf{q}}^2) dt^2 - \sum_{a=1}^p q_a^2(t) d\theta_a d\theta_a. \quad (14)$$

This representation shows that the change of  $\sigma^r$  by the new coordinates  $\theta_a(\sigma^r)$ , parametrizing  $\Sigma_p$ , makes the transformed metric independent of  $\sigma^r$ . In the  $\theta_a$  parametrization the above-mentioned Killing vector fields on  $\Sigma_p$  take the form of the derivatives  $\frac{\partial}{\partial \theta_a}$ . All that shows that  $\mathcal{U}$  invariant (hyper)surface (11) has zero curvature and is isometric to a flat  $p$ -dimensional torus  $S^1 \times S^1 \times \dots \times S^1$  (with  $p$  multipliers  $S^1$ ) at any fixed moment of time.

The canonical momentum components  $\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{q}_a} = \vec{\mathcal{P}} \frac{\partial \vec{x}}{\partial \dot{q}_a}$ , ( $a = 1, 2, \dots, p$ ),  $\boldsymbol{\pi} := (\pi_1, \dots, \pi_p)$  conjugate to the coordinates  $\mathbf{q}$  (11) may be presented in the explicit form as

$$\boldsymbol{\pi} = \mathcal{P}_0 \dot{\mathbf{q}}, \quad \mathcal{P}_0 = T \sqrt{\frac{|g|}{1 - \dot{\mathbf{q}}^2}} \quad (15)$$

after using (6) and the relations

$$\vec{x}^2 = \mathbf{q}^2, \quad \dot{\vec{x}}^2 = \dot{\mathbf{q}}^2, \quad g = \det\left(\sum_{a=1}^p q_a^2 \theta_{a,r} \theta_{a,s}\right). \quad (16)$$

Then the Hamiltonian density (4) in the  $(\mathbf{q}, \boldsymbol{\pi})$  phase space takes the form

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\boldsymbol{\pi}^2 + T^2|g|}, \quad \dot{\mathcal{P}}_0 = 0 \quad (17)$$

that matches with representation (15) for  $\mathcal{P}_0$  through the velocity  $\dot{\mathbf{q}}$

$$\mathcal{P}_0 = T \sqrt{\frac{|g|}{1 - \dot{\mathbf{q}}^2}}. \quad (18)$$

The corresponding Hamiltonian equations are transformed into the equations

$$\dot{\mathbf{q}} = \{H_0, \mathbf{q}\} = \frac{1}{\mathcal{P}_0} \boldsymbol{\pi}, \quad \dot{\boldsymbol{\pi}} = \{H_0, \boldsymbol{\pi}\}, \quad (19)$$

accompanied with the standard Poisson brackets and the Hamiltonian  $H_0$

$$H_0 = \int d^p \sigma \sqrt{\boldsymbol{\pi}^2 + T^2|g|}, \quad \{\pi_a, q_b\} = \delta_{ab}, \quad \{q_a, q_b\} = \{\pi_a, \pi_b\} = 0. \quad (20)$$

In the next section we study the equations of motion for the coordinates of the effective vector  $\mathbf{q}$ .

## 4 Equations of $\mathcal{U}$ invariant p-branes

To simplify the equations of the introduced  $\mathcal{U}$  invariant p-brane we choose the functions  $\theta_a(\sigma^r)$  in (11) to be linear. This can be done by additional gauge fixing for the residual gauge symmetry (9):  $\theta_a = \delta_{ar}\sigma^r$ , i.e. as <sup>3</sup>

$$\theta_1(\sigma^r) = \sigma^1, \quad \theta_2(\sigma^r) = \sigma^2, \quad \dots, \quad \theta_p(\sigma^r) = \sigma^p. \quad (21)$$

In the gauge (21) the anzats (11) and  $g_{rs}$  (13) are expressed as follows

$$\vec{x}^T(t) = (q_1 \cos \sigma^1, q_1 \sin \sigma^1, \dots, q_p \cos \sigma^p, q_p \sin \sigma^p), \quad (22)$$

$$g_{rs}(t) = q_r^2(t) \delta_{rs}, \quad g = (q_1 q_2 \dots q_p)^2 \quad (23)$$

with the diagonalized metric  $g_{rs}(t)$  depending only on the time  $t$ .

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<sup>3</sup>To cover the case of p-brane with windings one can fix the gauge conditions as  $\theta_a = n_a \delta_{ar} \sigma^r$ , where  $(n_1, n_2, \dots, n_p)$  are the integer numbers corresponding to the winding numbers on the circles  $0 \leq \sigma^r \leq 2\pi$  parametrized by  $\sigma^r$ .



As a result, the interval (14) and the inverse metric tensor  $g^{rs}$  reduced on  $\Sigma_p$  take the form

$$ds_p^2 = \sum_{r=1}^p q_r^2(t)(d\sigma^r)^2, \quad g^{rs}(t) = \frac{1}{q_r^2} \delta_{rs}. \quad (24)$$

Gauge (21) clarifies that coordinates  $\mathbf{q}(t) = (q_1, q_2, \dots, q_p)$  are the time dependent radii  $\mathbf{R}(t) = (R_1, R_2, \dots, R_p)$  of the flat  $p$ -torus  $\Sigma_p$ . In the gauge (21) the Hamiltonian density  $\mathcal{H}_0$  becomes independent of the  $p$ -torus parameters  $\sigma^r$  and reduces to a constant  $C$  chosen to be positive

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\pi^2 + T^2 \prod_{r=1}^p q_r^2} = C, \quad \dot{\mathcal{P}}_0 = \frac{\partial}{\partial \sigma^r} \mathcal{P}_0 = 0. \quad (25)$$

Equations (25) combined with (18) are equivalent to the initial data condition

$$\mathcal{P}_0 = C \quad \rightarrow \quad T \sqrt{\frac{(q_1 q_2 \dots q_p)^2}{1 - \dot{\mathbf{q}}^2}} = C. \quad (26)$$

Then Eqs. (6) for the vector  $\vec{x}$  are simplified to the form

$$\ddot{\vec{x}} - \left(\frac{T}{C}\right)^2 g g^{rs} \partial_{rs} \vec{x} = 0. \quad (27)$$

Let us take into account the relations following from (22) and (23)

$$g g^{rs} = \frac{\delta_{rs}}{q_r^2} \prod_{t=1}^p q_t^2, \quad \Delta^{(p)} \vec{x} = -\vec{x}, \quad (28)$$

where  $\Delta^{(p)} := \sum_{r=1}^p \partial_r^2$  is the Laplace operator. The use of (28) transforms the system (27), equivalent to the Hamiltonian equations (19), into the algorithmic chain of  $p$  nonlinear equations for the  $\mathbf{q}(t)$  components  $q_1, q_2, \dots, q_p$

$$\ddot{q}_r + \left(\frac{T}{C}\right)^2 (q_1 \dots q_{r-1} q_{r+1} \dots q_p)^2 q_r = 0, \quad (29)$$

where the component index  $r$  runs from 1 to  $p$ .

Multiplication of the  $r$ -th equation of the system (29) by  $q_r$  and subsequent summing over  $r$  results in the first integral of Eqs. (29)

$$\dot{\mathbf{q}}^2 + \left(\frac{T}{C}\right)^2 (q_1 q_2 \dots q_p)^2 = c \quad (30)$$

which coincides with the initial data (26) if the integration constant  $c = 1$ .

It is easily seen that Eqs. (29) may be presented in a condensed form as

$$C\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}, \quad (31)$$

where the elastic energy density  $V(\mathbf{q})$  turns out to be proportional to the determinant  $g$  of the  $\mathcal{U}$  invariant (hyper)surface of  $p$ -brane

$$V(\mathbf{q}) = \frac{T^2}{2C}g \equiv \frac{T^2}{2C}(q_1 \dots q_p)^2. \quad (32)$$

Thus, we see that the dynamics of a toroidal  $p$ -brane is handled by anharmonic elastic potential (32) positively defined for  $C > 0$ .

## 5 $U(1) \times U(1) \times \dots \times U(1)$ invariant $p$ -branes and $p$ -dimensional anharmonic oscillators

The derived equations (29) may be generated by a Hamiltonian  $H$  free from the square root present in  $H_0$  (20). Such a possibility is a consequence of our fixing the residual gauge symmetry that reduces  $\mathcal{P}_0$  to the constant  $C$ .

As a result,  $C$  can be used to write the Hamiltonian density free from square root

$$\mathcal{H} = \frac{\mathcal{H}_0^2}{2C} = C/2.$$

Hamiltonian  $H$  and PB's associated with the density  $\mathcal{H}$  are

$$H := \int d^p \sigma \mathcal{H}, \quad \mathcal{H} = \frac{1}{2C}(\boldsymbol{\pi}^2 + T^2 \prod_{a=1}^p q_a^2), \quad (33)$$

$$\{\pi_a, q_b\} = \delta_{ab}, \quad \{q_a, q_b\} = 0, \quad \{\pi_a, \pi_b\} = 0.$$

The Hamiltonian (33) generates Eqs. (29) and describes  $p$ -dimensional anharmonic oscillator without quadratic terms associated with the harmonic oscillator. The set of Hamiltonians (33) contains potential energy quartic in  $\mathbf{q}$  for  $p = 2$  or presented by higher degree monomials for  $p > 2$ .

A physical interpretation of nonlinear system (29) in terms of harmonic oscillator shows that each of the coordinates  $q_r(t)$ , undergoing the force  $F_r(t)$ ,

evolves with the instant cyclic "frequency"  $\omega_r(t)$  proportional to the product of all the remaining coordinates at any fixed moment  $t$

$$\omega_r(t) = \frac{T}{C} |q_1 \dots q_{r-1} q_{r+1} \dots q_p|, \quad r = (1, 2, \dots, p), \quad (34)$$

as it follows from (29). These "frequencies"  $\omega_r$  cannot be infinitely large because of the initial data constraint (30) with  $c = 1$

$$\sqrt{1 - \dot{\mathbf{q}}^2} = \frac{T}{C} |q_1 q_2 \dots q_p|, \quad (35)$$

which manifests the conservation of the energy density and shows that

$$0 \leq |\dot{\mathbf{q}}| \leq 1, \quad 0 \leq \frac{T}{C} |q_1 q_2 \dots q_p| \leq 1. \quad (36)$$

Eqs. (36) means that the velocity  $|\dot{\mathbf{q}}|$  grows if the  $p$ -brane (hyper)volume  $\sim |q_1 q_2 \dots q_p|$  diminishes and reaches the velocity of light ( $|\dot{\mathbf{q}}| = 1$ ) while the (hyper)volume vanishes. The minimal velocity  $\dot{\mathbf{q}} = 0$  corresponds to the maximal (hyper)volume  $\sim |q_1 q_2 \dots q_p|$  equal to  $C/T$ . This explains the finiteness of  $\omega_r$

$$\omega_r(t) = \frac{\sqrt{1 - \dot{\mathbf{q}}^2}}{|q_r|}. \quad (37)$$

In the case of tensionless p-branes [9, 11] Eqs. (29) reduce to the equations

$$T = 0 : \Rightarrow \quad \ddot{\mathbf{q}} = 0, \quad |\dot{\mathbf{q}}| = 1, \quad (38)$$

similar to the integrable equation of free massless particle in the effective  $p$ -dimensional  $\mathbf{q}$ -space.

In general, Eqs. (29) are complicated owing to a monomial entanglement of  $q_r$  in potential  $V$  (33). However, there is a solvable case discussed below.

## 6 Elliptic and hyperelliptic solutions

In the degenerate case characterized by the coincidence of all the components  $q_1 = q_2 = \dots = q_p \equiv q$  Eqs. (29) are reduced to one nonlinear differential equation

$$\ddot{q} + \left(\frac{T}{C}\right)^2 q^{(2p-1)} = 0 \quad (39)$$

equivalent to its first integral (35) expressing the energy conservation

$$p\dot{q}^2 + \left(\frac{T}{C}\right)^2 q^{2p} = 1. \quad (40)$$

After the change of  $q$  by the new variable  $y = \Omega^{\frac{1}{p}} \sqrt{p} q$ , with  $\Omega := \frac{T}{C} p^{-\frac{p}{2}}$ , equation (40) takes the form

$$\left(\frac{dy}{d\tilde{t}}\right)^2 = \frac{1}{2}(1 - y^p)(1 + y^p), \quad (41)$$

where a new time variable  $\tilde{t} := \sqrt{2}\Omega^{\frac{1}{p}}t$  is introduced.

In the membrane case ( $p = 2$ ) Eq.(41) coincides with the equation defining the Jacobi elliptic cosine  $cn(x; k)$

$$\left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 + k^2 y^2), \quad (42)$$

if the elliptic modulus  $k = \frac{1}{\sqrt{2}}$ . Thus,  $y(t) = cn(\sqrt{2\omega}t; \frac{1}{\sqrt{2}})$  with  $2\omega = T/C$ .

After using the relation  $q \equiv y/\sqrt{2\omega}$  we obtain the elliptic solution for the desired coordinate  $q(t)$ ,

$$q(t) = \frac{1}{\sqrt{2\omega}} cn(\sqrt{2\omega}(t + t_0); \frac{1}{\sqrt{2}}) \equiv \sqrt{\frac{C}{T}} cn\left(\sqrt{\frac{T}{C}}(t + t_0); \frac{1}{\sqrt{2}}\right) \quad (43)$$

that is similar to the elliptic solution for  $U(1)$  invariant membrane in  $D = 5$  earlier obtained in [19].

If the initial velocity  $\dot{q}(t_0) > 0$  then the solution (43) describes an expanding torus which reaches the maximal size  $q_{max} = \sqrt{\frac{C}{T}}$  at some moment  $t$  and then contracts to a point in a finite time  $\mathbf{K}(1/\sqrt{2})\sqrt{\frac{C}{T}}$  (where  $\mathbf{K}(1/\sqrt{2}) = 1.8451$ ) is the quarter period of the elliptic cosine).

An explicit equation of the surface  $\Sigma_2(t)$  of the contracting torus (43) is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{4C}{T} cn\left(\sqrt{\frac{T}{C}}(t + t_0), \frac{1}{\sqrt{2}}\right)^2, \quad x_1 x_4 = x_2 x_3.$$

For the case  $p > 2$  integration of Eq.(41) results in the solution

$$\tilde{t} = \pm \sqrt{2} \int \frac{dy}{\sqrt{1 - y^{2p}}} + const \quad (44)$$

that contains hyperelliptic integral and defines an implicit dependence of  $q$  on time. Thus, the general solution of Eq. (40) is expressed in terms of hyperelliptic functions generalizing elliptic functions.

The variable change  $z = y^{2p}$  transforms the solution (44) into the integral

$$\tilde{t} - \tilde{t}_0 = \pm \frac{1}{\sqrt{2p}} \int_0^{z^{\frac{1}{2p}}} dz z^{(\frac{1}{2p}-1)} (1-z)^{-\frac{1}{2}} \quad (45)$$

similar to the integral discussed in [16]. The use of representation (45) allows to find the contraction time  $\Delta\tilde{t}_c$  of the degenerate  $p$ -torus from its maximal size defined by the coinciding radii value  $q_{max} = (\frac{C}{T})^{\frac{1}{p}}$  to  $q_{min} = 0$ .

This time turns out to be proportional to the well-known integral

$$\Delta\tilde{t}_c = \frac{1}{\sqrt{2p}} \int_0^1 dz z^{(\frac{1}{2p}-1)} (1-z)^{\frac{1}{2}-1} = \frac{1}{\sqrt{2p}} B\left(\frac{1}{2p}, \frac{1}{2}\right)$$

which defines the Euler beta function  $B(\frac{1}{2p}, \frac{1}{2})$ .

Coming back to the original time  $t$  and taking into account that  $C = 2E$  we find the desired contraction time

$$\Delta t_c \equiv \frac{1}{\sqrt{2}} \Omega^{-\frac{1}{p}} \Delta\tilde{t}_c = \frac{1}{2\sqrt{p}} \left(\frac{2E}{T}\right)^{\frac{1}{p}} B\left(\frac{1}{2p}, \frac{1}{2}\right) \quad (46)$$

as a function of the  $p$ -brane dimension  $p$  and its energy density  $E$ .

So, the case of degenerate toroidal  $p$ -brane with coinciding radii is proved to be integrable and revealing the connection of the  $p$ -brane equations with the (hyper)elliptic and Euler beta functions.

## 7 Anharmonic deformation of the Hooke law

To add to understanding of the physics described by the  $\mathcal{U}$  invariant  $p$ -branes one should analyse Eqs. (31). As is seen they are representable as the equations of motion of general elastic medium with the mass density  $\rho$

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k}, \quad (47)$$

where  $\ddot{u}_i$  and  $\sigma_{ik}$  are the medium acceleration and its stress tensor [24]. This becomes apparent after using the symmetric stress tensor  $\sigma_{rs}$

$$\sigma_{rs} := -p \delta_{rs}, \quad p := \frac{T^2}{2C} g \equiv \frac{T^2}{2C} \prod_{s=1}^p q_s^2, \quad (48)$$

where  $p = V$  is an isotropic pressure per unit (hyper)area of  $p$ -brane (hyper)volume. Then Eqs. (31) take the form similar to Eqs. (47)

$$C\ddot{q}_r = -\frac{\partial[\delta_{rs}T^2(2C)^{-1}(q_1q_2\dots q_p)^2]}{\partial q_s} \quad (49)$$

with the constant  $C$  substituted for the mass density  $\rho$ . The constant  $C$  is the energy density  $\mathcal{P}_0 = T|q_1q_2\dots q_p|/\sqrt{1-\dot{\mathbf{q}}^2}$  (26) of the toroidal  $p$ -brane taken in the fixed gauge for the worldvolume diffeomorphisms. From physical point of view the replacement  $\rho \rightarrow \mathcal{P}_0$  is typical of the standard transition to relativistic mechanics, e.g. when the rest energy  $m$  of a non-relativistic particle is replaced by its relativistic energy, i.e.  $m \rightarrow m/\sqrt{1-\dot{\mathbf{x}}^2}$  (where the velocity of light is absent in accordance with our agreements).

However, Eqs. (49) have no explicit relativistic covariant form in contrast to the original equations (1). This breakdown is a result of the choice of both the gauge fixing and the ansatz (22). Such a non-relativistic formulation of the toroidal  $p$ -brane dynamics gives a reason to raise the question about the possibility of the existence of non-relativistic elastic media described by equations (49) in solid state physics. To answer this question let us note that the  $p$ -brane medium pressure  $p$  (48) is created by the internal force  $F_r$

$$F_r(t) = -\frac{\partial V}{\partial q_r} \equiv -\frac{T^2}{C}(q_1\dots q_{r-1}q_{r+1}\dots q_p)^2q_r \quad (50)$$

produced by the elastic potential  $V$  (32). Relations (50) manifest the breakdown of linear Hooke's law and its replacement by the nonlinear law. They may be presented in a vector-like form

$$\mathbf{F}(t) = -\frac{\partial V}{\partial \mathbf{q}} \equiv -\frac{T^2}{C}|\mathbf{q}|^{2(p-1)}\mathbf{q} \quad (51)$$

for the case of integrable degenerate toroidal  $p$ -brane, discussed in the previous section. Similar anharmonic generalization of the harmonic Hooke law was earlier revealed in liquid crystals such as 2d and 3d smectics  $\mathcal{A}$ , resulting in the study of elastic critical points, anisotropic scaling behaviour and renormalizations of 3d smectic elastic constants.

So, studying the *branes*  $\leftrightarrow$  *smectics* correspondence induced by the anharmonic elasticity, characterizing both toroidal  $p$ -branes and liquid crystals, may give raise to a new insight in the brane physics. The well-known general intertwines of the statistical mechanics and field theory may also hint in behalf of such a studying.

## 8 Discussion

Proposed is a new  $U(1) \times U(1) \times \dots \times U(1)$  invariant ansatz describing the  $p$ -brane hypersurfaces  $\Sigma_p$  embedded in the  $D = (2p + 1)$ -dimensional Minkowski space. The compact hypersurface  $\Sigma_p$  is shown to be isometric to a flat  $p$ -dimensional torus with time-dependent radii. The equations of toroidal  $p$ -brane are derived and reduced to an algorithmic chain of entangled equations associated with  $p$ -dimensional anharmonic oscillator. Constructed is the Hamiltonian of  $p$ -dimensional anharmonic elasticity described by the monomial potential of the degree  $2p$  (with  $p = 2, 3, \dots, (D - 1)/2$ ). These equations are proved to be integrable in the case of the degenerate  $p$ -torus characterized by equal radii. The obtained general solutions describe contracting  $p$ -tori and are presented by the elliptic cosine for  $p=2$  or hyperelliptic functions for higher  $p > 2$ . The exact formula for the contraction time of the  $p$ -branes is found. These results provide an example of admissible anharmonic potentials, associated with branes and, in particular, with the 5-brane of  $D = 11$  M/string theory. They also demonstrate a type of possible complications accompanying the transition from strings to branes. As a result of the complications, the space of trigonometric functions associated with the string dynamics undergoes deformation controlled, in particular, by the values of the moduli of (hyper)elliptic functions.

On the other hand, taking into account the geometric consideration of strings and branes as (hyper)surfaces embedded in the Minkowski space-time one may try to construct the general solution of brane equations assuming the existence of corresponding Backlund transformations. Then the application of such conceivable transformations to the (hyper)elliptic solutions would generate an infinite number of conservation laws by analogy with the case of strings or two-dimensional integrable sigma models [25]. This assumption is substantiated by the Regge-Lund geometric approach originally applied to strings [26]. In the geometric approach the string dynamics is reformulated into the consideration of the string worldsheet as a two-dimensional surface embedded in the four-dimensional Minkowski space-time. The projection of the string worldsheet on a three-dimensional hypersurface of a constant time studied in [26] resulted in the classical problem of surfaces as manifolds embedded in the three-dimensional Euclidean space. The Gauss-Codazzi equations fixing the conditions for the embedding were reduced there to the sine-Gordon equation. Further, the Gauss-Weingarten equations (the first order linear differential equations for the tangent and normal vectors) were

proved to be presented in the form of the linear equations just associated with the sine-Gordon equation in the inverse scattering method [27], [28]. Similar results follow from the generalization of the Regge-Lund approach for strings evolving in higher dimensional Minkowski spaces [29], [30], [31], [32]. The Maurer-Cartan structure equations were used in [32] to describe the string worldsheet in higher dimensional spaces. These equations play the role of integrability conditions for the linear equations describing a repere moving along the string worldsheet embedded into  $D$ -dimensional Minkowski space. As a result, the Nambu-Goto string was shown to describe a closed sector of the two-dimensional  $SO(1, 1) \times SO(D - 2)$  gauge model with the Maurer-Cartan equations reduced to the nonlinear chain [32] of PDE's for the  $(D - 2)$  sectorial curvatures of the string worldsheet. This approach may be applied for the consideration of  $p$ -branes as embedded (hyper)surfaces. In the case of  $p$ -brane embedded in  $D$ -dimensional Minkowski space-time the string local group  $SO(1, 1) \times SO(D - 2)$  has to be substituted by the local  $SO(1, p) \times SO(D - p - 1)$  group attached to the  $p$ -brane worldvolume. Then the time-dependent (hyper)elliptic solutions considered as particular solutions in associated  $SO(1, D - 1)/SO(1, p) \times SO(D - p - 1)$  sigma models would be used for the construction of desired general solution with help of above discussed Backlund transformations.

Studying the physics associated with toroidal  $p$ -branes we find that they may be treated as elastic media described by anharmonic generalizations of Hooke's law and subjected to time-dependent isotropic pressure. As mentioned in the previous section similar type of anharmonic elasticity was earlier discovered in smectics  $\mathcal{A}$ , and therefore these liquid crystals can be considered as media modelling some properties of toroidal  $p$ -branes. Then one can try to apply the physics of smectics to the branes of M-theory to get more information on the brane physics.

Another issue which deserves an attention is the study of toroidal  $p$ -branes in curved space-time and, in particular, in Anti de Sitter space-time for further verification of the AdS/CFT duality conjection.

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